NOTRE-DAME UNIVERSITY

Faculty of Engineering

Department of Electrical, Computer and Communications Engineering

FALL 2016

EEN 340: SIGNALS AND SYSTEMS

MIDTERM EXAMINATION - I -

SOLUTION KEY

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Problem 1: Understanding Fundamental Concepts (25 Points)

This problem is composed of four parts revolving around four different topics. They are as follows.

PART A: Orthogonal Functions (5 Points)

Consider the functions $f_1(t) = A\sin(4\pi t)$ and $f_2(t) = B\sin(2\pi t)$. Prove that these two functions are orthogonal over the interval $\left[-\frac{1}{2}; \frac{1}{2}\right]$. Show all the proof details in the box below.

Solution:

In order to show the orthogonality of the functions $f_1(t)$ and $f_2(t)$, evaluate the following integral:

$$
\begin{aligned}
I &= \int_{-\frac{1}{2}}^{+\frac{1}{2}} f_1(t) \cdot f_2^*(t)\mathrm{d}t = (A \cdot B) \int_{-\frac{1}{2}}^{+\frac{1}{2}} \sin(4\pi t)\sin(2\pi t)\mathrm{d}t \\
&= \frac{A \cdot B}{2} \left[\int_{-\frac{1}{2}}^{+\frac{1}{2}} \cos(2\pi t)\mathrm{d}t - \int_{-\frac{1}{2}}^{+\frac{1}{2}} \cos(6\pi t)\mathrm{d}t \right] \\
&= \frac{A \cdot B}{2} \frac{1}{2\pi} \left[\sin\left(2\pi \frac{1}{2}\right) - \sin\left(-2\pi \frac{1}{2}\right) \right] - \frac{A \cdot B}{2} \frac{1}{6\pi} \left[\sin\left(6\pi \frac{1}{2}\right) - \sin\left(-6\pi \frac{1}{2}\right) \right] \\
&= 0
\end{aligned}
$$

PART B: Properties of the Unit Impulse Function (5 Points)

Evaluate the following two integrals.

$$\int_{-\infty}^{+\infty} \sin(t-1)\delta(2t-4)\mathrm{d}t$$

Solution:

In the above integral, note that the function $\delta(2t-4) \neq 0$ whenever $2t - 4 = 0 \Rightarrow t = 2$. As such $\sin(t-1)\delta(2t-4) = \sin(2-1)\delta(2t-4) = \sin(1)\delta(2t-4)$. Consequently:

$$\int_{-\infty}^{+\infty} \sin(t-1)\delta(2t-4)\mathrm{d}t = \int_{-\infty}^{+\infty} \sin(1)\delta(2t-4)\mathrm{d}t = \sin(1)\int_{-\infty}^{+\infty} \delta(2t-4)\mathrm{d}t$$

Applying a change of variable, let $\tau = 2t - 4 \Rightarrow \tau \in [-\infty; +\infty]$ and $\mathrm{d}\tau = 2\mathrm{d}t$. As such:

$$\sin(1)\int_{-\infty}^{+\infty} \delta(2t-4)\mathrm{d}t = \sin(1)\int_{-\infty}^{+\infty} \delta(\tau)\frac{\mathrm{d}\tau}{2} = \frac{1}{2}\sin(1) = 0.4207$$

$$\int_{-\infty}^{+\infty} \cos\left(2t - \frac{\pi}{2}\right)\delta\left(t - \frac{\pi}{4}\right)\mathrm{d}t$$

Solution:

Observe that:

$$\cos\left(2t - \frac{\pi}{2}\right)\delta\left(t - \frac{\pi}{4}\right) = \cos\left(2\frac{\pi}{4} - \frac{\pi}{2}\right)\delta\left(t - \frac{\pi}{4}\right) = \cos(0)\,\delta\left(t - \frac{\pi}{4}\right) = \delta\left(t - \frac{\pi}{4}\right)$$

As such:

$$\int_{-\infty}^{+\infty} \cos\left(2t - \frac{\pi}{2}\right)\delta\left(t - \frac{\pi}{4}\right)\mathrm{d}t = \int_{-\infty}^{+\infty} \delta\left(t - \frac{\pi}{4}\right)\mathrm{d}t = 1$$

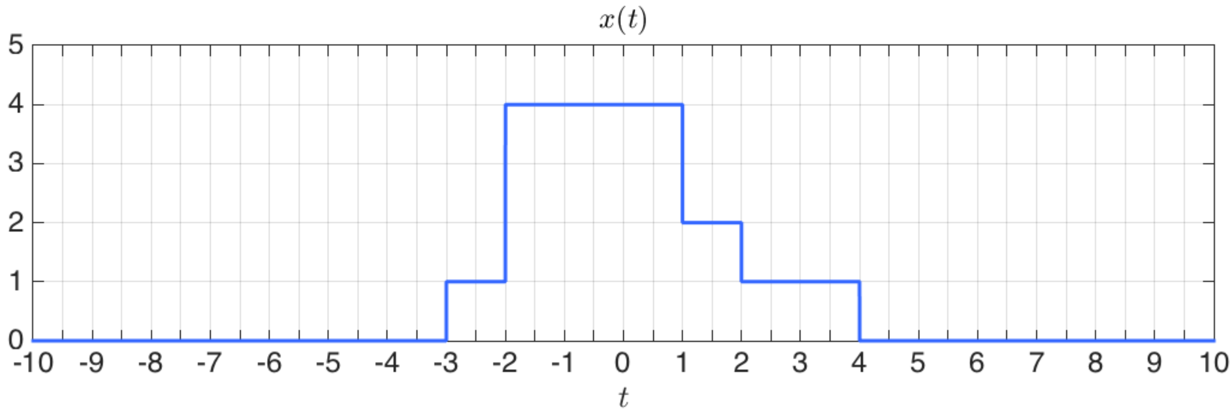
PART C: Plotting Signals (10 Points)

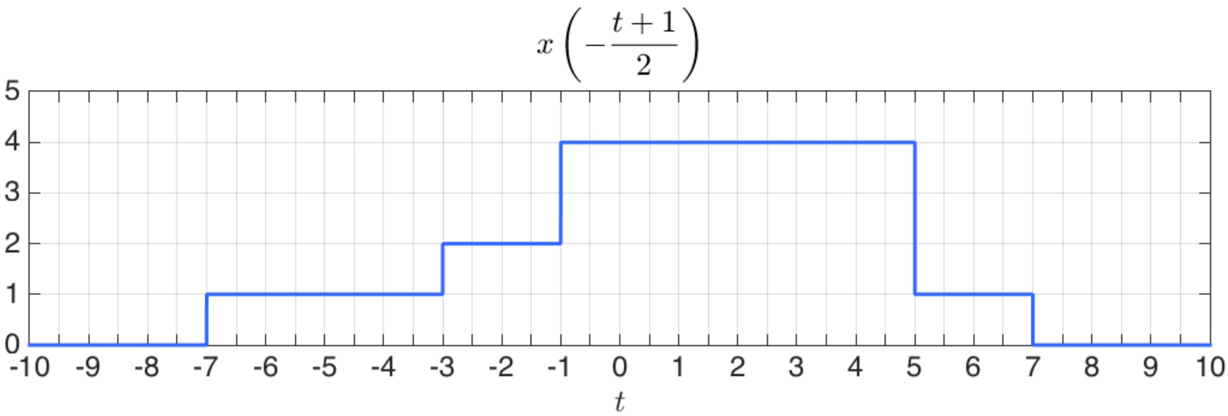
Consider the signal:

$$x(t) = 3u(t+2) + u(t+3) - u(t-4) - u(t-2) - 2u(t-1)$$

Use the below two graphs respectively to:

1. Plot the signal $x(t)$. **(2 Points)**
2. Plot the signal $x\left(-\frac{t+1}{2}\right)$. **(8 Points)**





PART D: Systems Characteristics (5 Points)

Consider a system having an input $x(t)$ and an output $y(t)$ such that:

$$y(t) = \int_t^{t+1} x(\tau - \alpha)\mathrm{d}\tau \text{ , for } \alpha \in \mathbb{R}$$

Answer the following questions (show all your work in order to have credit):

1. Study the linearity and time invariance of the above-described system. **(2 Points)**
2. Determine the values of α for which the above-described system will be causal. **(3 Points)**

Solution:

1. Being by verifying the linearity of the system. For this reason, consider two inputs $x_1(t)$ and $x_2(t)$ together with their respective outputs $y_1(t)$ and $y_2(t)$ such that:

$$x_1(t) \to y_1(t) = \int_t^{t+1} x_1(\tau - \alpha)\mathrm{d}\tau$$
$$x_2(t) \to y_2(t) = \int_t^{t+1} x_2(\tau - \alpha)\mathrm{d}\tau$$

Now, consider the input $\chi(t) = \lambda x_1(t) + \mu x_2(t)$ where $(\lambda, \mu) \in \mathbb{R}$. Denote by $\gamma(t)$ the output corresponding to $\chi(t)$. It is given by:

$$\gamma(t) = \int_t^{t+1} \chi(\tau - \alpha)\mathrm{d}\tau = \int_t^{t+1} [\lambda x_1(\tau - \alpha) + \mu x_2(\tau - \alpha)]\mathrm{d}\tau$$
$$= \lambda \int_t^{t+1} x_1(\tau - \alpha)\mathrm{d}\tau + \mu \int_t^{t+1} x_2(\tau - \alpha)\mathrm{d}\tau = \lambda y_1(t) + \mu y_2(t) \Rightarrow \textbf{The system is linear.}$$

Next, verify the time-invariance of the system. For this reason, consider an input $x(t - t_0)$. the corresponding output will be:

$$z(t) = \int_{t-t_0}^{t-t_0+1} x(\tau - \alpha)\mathrm{d}\tau = \int_{t-t_0}^{(t+1)-t_0} x(\tau - \alpha)\mathrm{d}\tau = y(t - t_0) \Rightarrow \textbf{The system is time-invariant.}$$

2. For causality, it is required that $y(t_0)$ be only dependent on values of $x(t)$ such as $t \leq t_0$. Accordingly:

$$t_0 + 1 - \alpha \leq t_0 \Rightarrow \alpha \geq 1$$

Problem 2: Convolution (25 Points)

Consider a system having an impulse response $h(t)$. This system admits as input a signal $x(t)$ and outputs a corresponding signal $y(t)$. Assuming that:

$$x(t) = \text{ramp}(-t)u(t+1) + \text{rect}\left(t - \frac{1}{2}\right) \qquad \text{and} \qquad h(t) = \text{rect}(t)$$

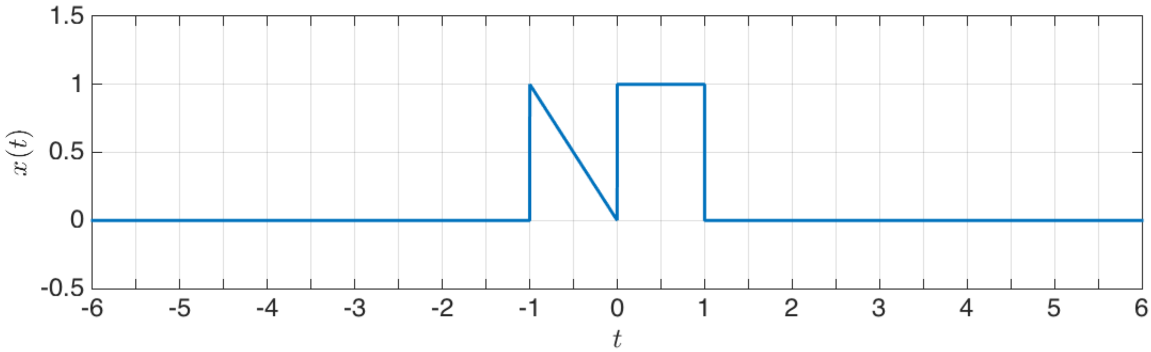
Derive an expression of the output $y(t)$. Show all derivation details with appropriate illustrations.

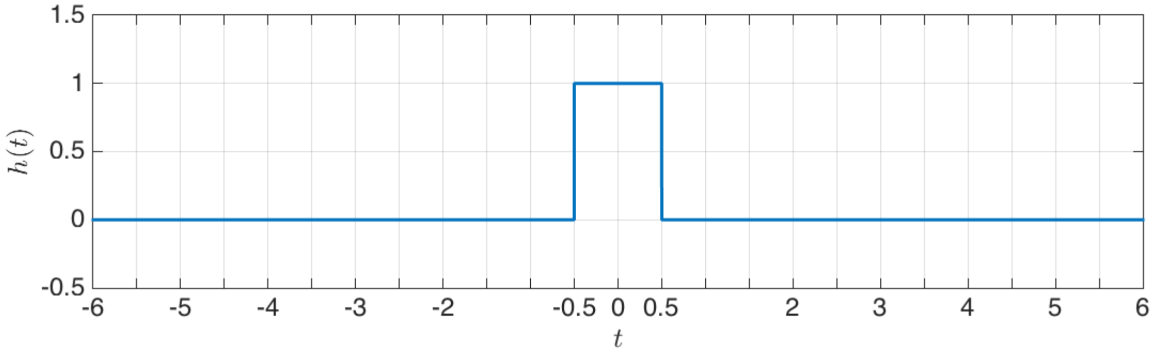
Solution:

This problem consists of finding an expression for $y(t)$ such that:

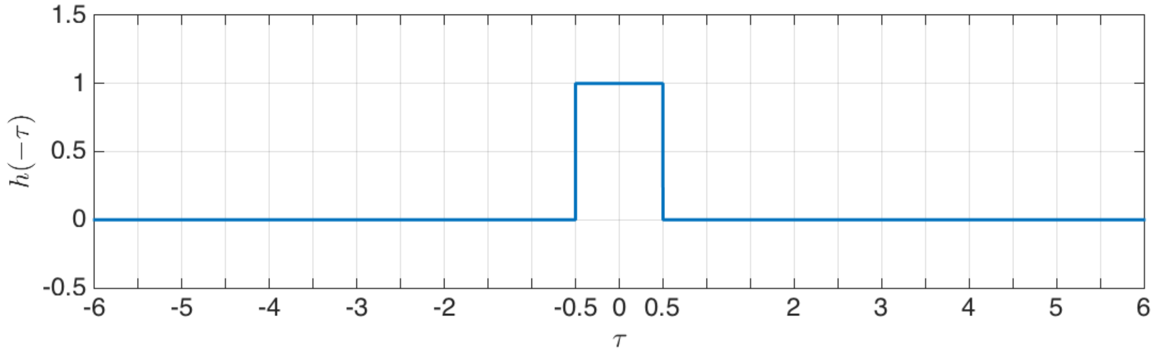
$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot h(t-\tau) \mathrm{d}\tau$$

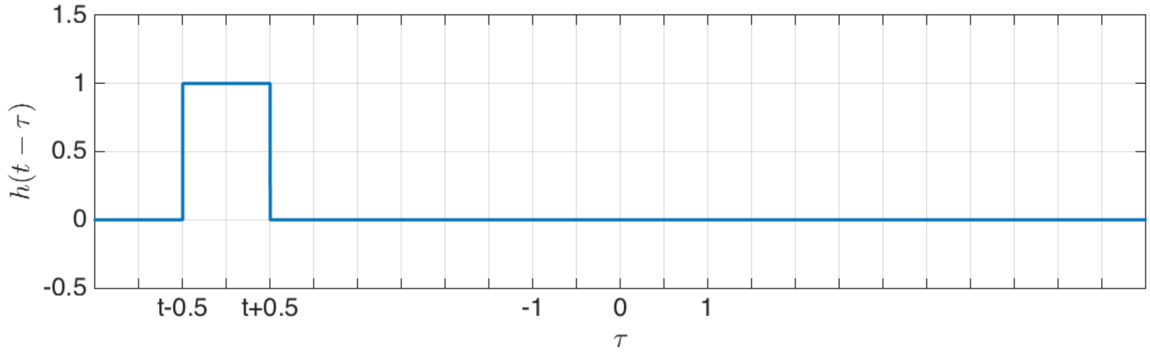
As a starting point, the evaluation of the above integral is performed in light of a clear visualization of the signals. For this purpose, the signals $x(t)$ and $h(t)$ are first plotted as follows.





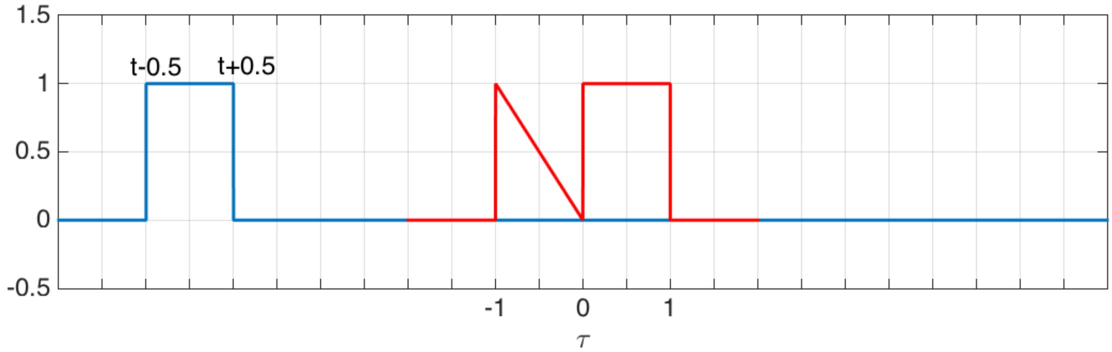
Now, after expressing the signals as a function of a dummy variable τ, one of the signals will be reflected with respect to the y-axis and then shifted by t. The chosen signal for reflection and shifting is $h(t)$. Note that since $h(\tau)$ is even, then its reflected version is identical to its original counterpart.





This being done, it is now the time to start shifting $h(t-\tau)$ and consider the different resulting overlaps with $x(\tau)$ as follows.

Case 1:



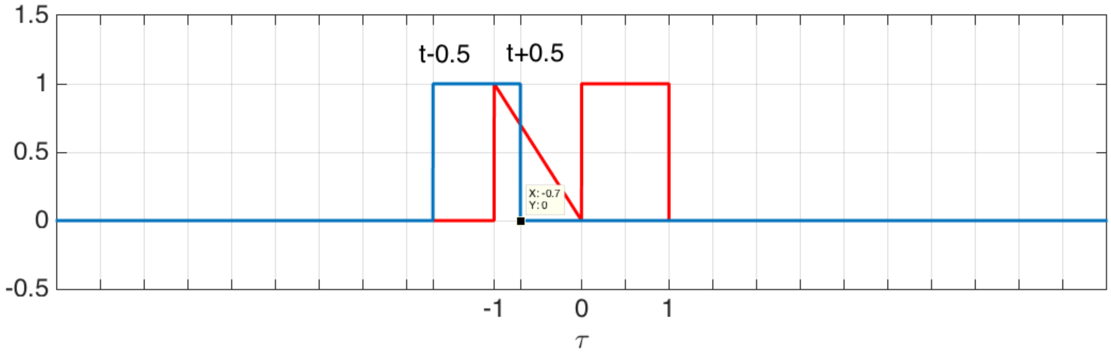
In this case:

$$t + \frac{1}{2} \leq -1 \Rightarrow t \leq -\frac{3}{2}$$

There exists no overlap between $x(\tau)$ and $h(t-\tau)$. Accordingly:

$$x(\tau) \cdot h(t-\tau) = 0 \Rightarrow y(t) = 0$$

Case 2:



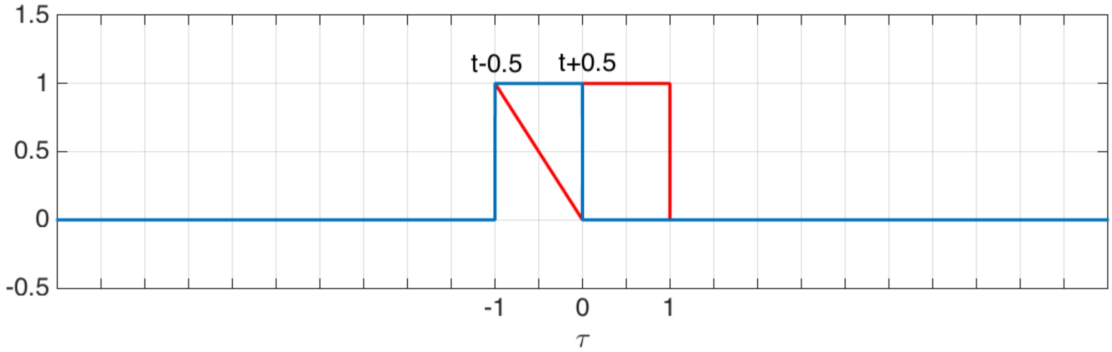
In this case:

$$t - \frac{1}{2} < -1 < t + \frac{1}{2} \Rightarrow -\frac{3}{2} < t < -\frac{1}{2}$$

There exists an overlap between $x(\tau)$ and $h(t-\tau)$ from -1 to $t + \frac{1}{2}$. Accordingly:

$$y(t) = \int_{-1}^{t+\frac{1}{2}} (-\tau \cdot 1) \mathrm{d}\tau = -\frac{\tau^2}{2}\Big|_{-1}^{t+\frac{1}{2}} = -\frac{1}{2}\left(t + 1/2\right)^2 + \frac{1}{2} = -\frac{(t^2+t)}{2} + \frac{3}{8}$$

Case 3:



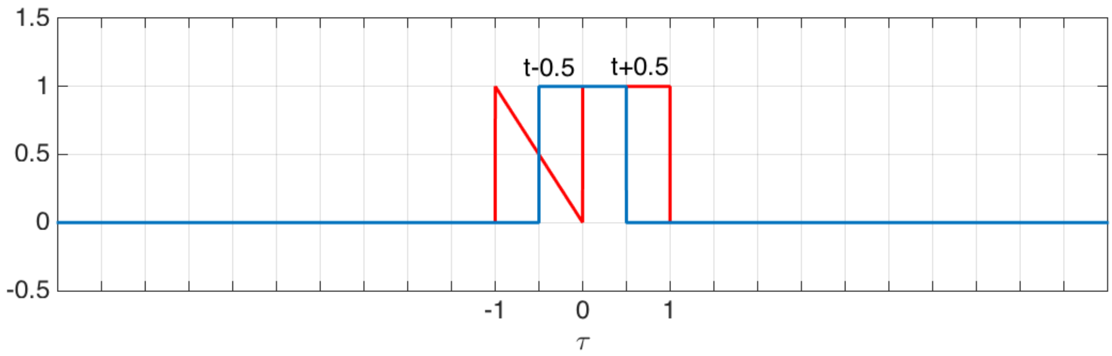
In this case:

$$t - \frac{1}{2} = -1 \quad \text{and} \quad t + \frac{1}{2} = 0 \Rightarrow t = -\frac{1}{2}$$

There exists an overlap between $x(\tau)$ and $h(t-\tau)$ from -1 to 0. Accordingly:

$$y(t) = \int_{-1}^{0} (-\tau \cdot 1) \mathrm{d}\tau = -\frac{\tau^2}{2}\Big|_{-1}^{0} = \frac{1}{2}$$

Case 4:



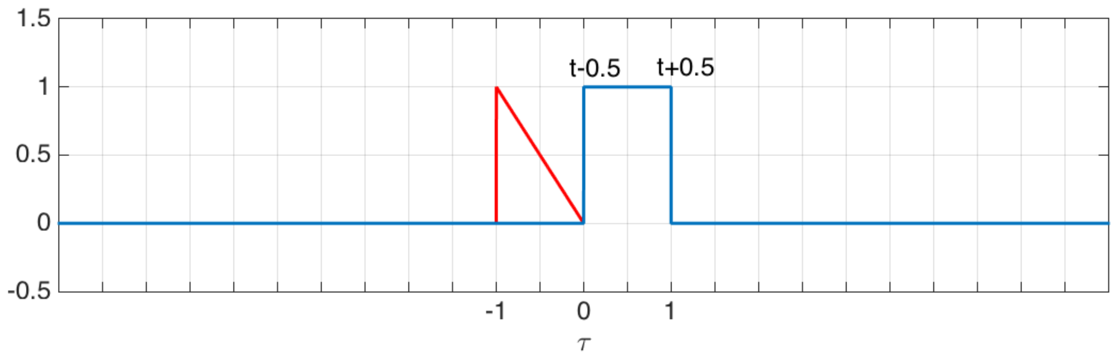
In this case:

$$t - \frac{1}{2} < 0 < t + \frac{1}{2} \Rightarrow -\frac{1}{2} < t < \frac{1}{2}$$

There exists an overlap between $x(\tau)$ and $h(t-\tau)$ from $t-\frac{1}{2}$ to $t+\frac{1}{2}$. However, it is important to observe that the product $x(\tau) \cdot h(t-\tau)$ has two different values respectively in the intervals $\left[t-\frac{1}{2}; 0\right]$ and $\left[0; t+\frac{1}{2}\right]$. Accordingly:

$$y(t) = \int_{t-\frac{1}{2}}^{0} (-\tau \cdot 1) \mathrm{d}\tau + \int_{0}^{t+\frac{1}{2}} (1 \cdot 1) \mathrm{d}\tau = -\frac{\tau^2}{2}\bigg|_{t-\frac{1}{2}}^{0} + \tau\bigg|_{0}^{t+\frac{1}{2}} = \frac{1}{2}\left(t - \frac{1}{2}\right)^2 + t + \frac{1}{2} = \frac{t^2 + t}{2} + \frac{5}{8}$$

Case 5:



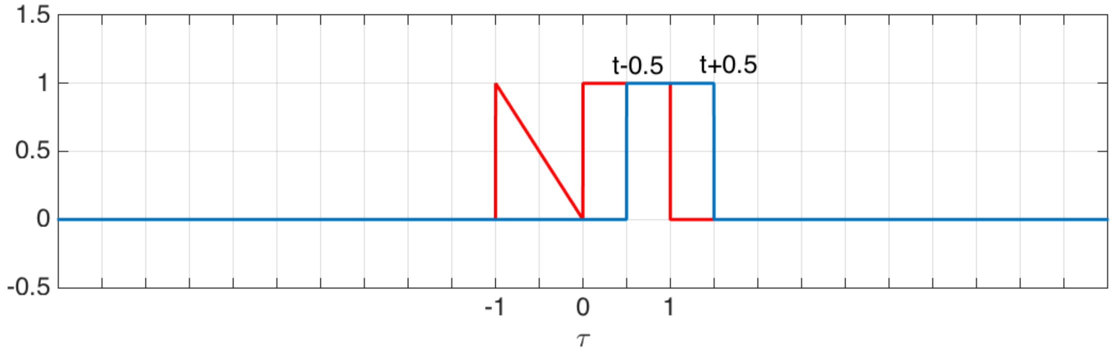
In this case:

$$t - \frac{1}{2} = 0 \quad \text{and} \quad t + \frac{1}{2} = 1 \Rightarrow t = \frac{1}{2}$$

There exists an overlap between $x(\tau)$ and $h(t-\tau)$ from 0 to 1. Accordingly:

$$y(t) = \int_{0}^{1} (1 \cdot 1) \mathrm{d}\tau = \tau\bigg|_{0}^{1} = 1$$

Case 6:



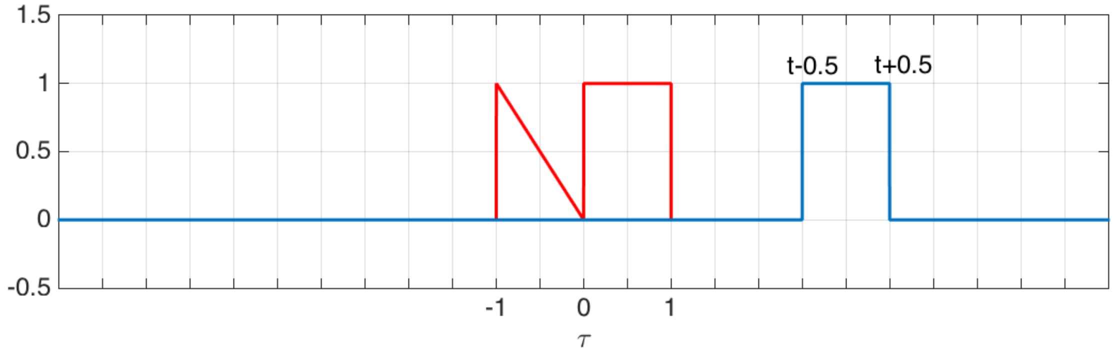
In this case:

$$t - \frac{1}{2} < 1 < t + \frac{1}{2} = 1 \Rightarrow \frac{1}{2} < t < \frac{3}{2}$$

There exists an overlap between $x(\tau)$ and $h(t-\tau)$ from $t - \frac{1}{2}$ to 1. Accordingly:

$$y(t) = \int_{t-\frac{1}{2}}^{1} (1 \cdot 1) \mathrm{d}\tau = \tau \Big|_{t-\frac{1}{2}}^{1} = 1 - t + \frac{1}{2} = -t + \frac{3}{2}$$

Case 7:



In this case:

$$t - \frac{1}{2} > 1 \Rightarrow t > \frac{3}{2}$$

There exists no overlap between $x(\tau)$ and $h(t-\tau)$. Accordingly:

$$x(\tau) \cdot h(t-\tau) = 0 \Rightarrow y(t) = 0$$

Finally, to this end, grouping all of the above cases together, leads to having:

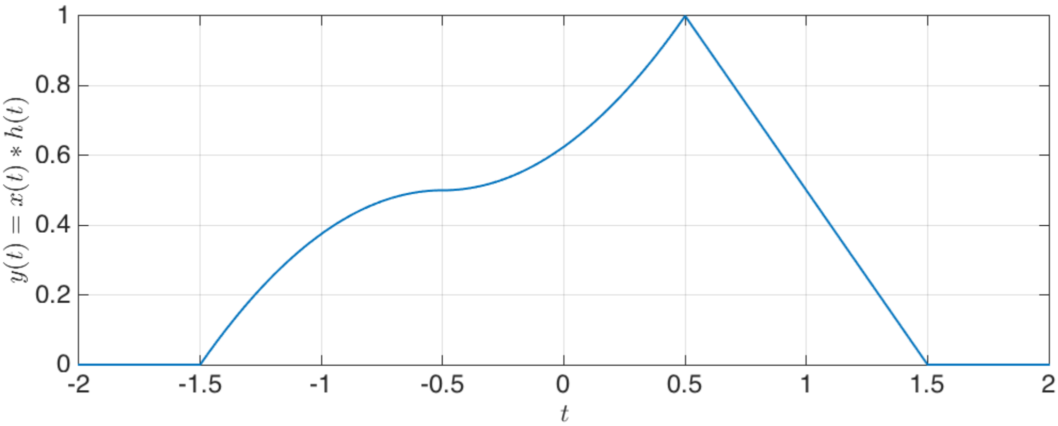
$$y(t) = \begin{cases} 0 & \text{, for } t < \frac{3}{2} \\ -\frac{(t^2+t)}{2} + \frac{3}{8} & \text{, for } -\frac{3}{2} \leq t < -\frac{1}{2} \\ \frac{1}{2} & \text{, for } t = -\frac{1}{2} \\ \frac{t^2+t}{2} + \frac{5}{8} & \text{, for } -\frac{1}{2} < t < \frac{1}{2} \\ 1 & \text{, for } t = \frac{1}{2} \\ -t + \frac{3}{2} & \text{, for } \frac{1}{2} < t \leq \frac{3}{2} \\ 0 & \text{, for } t > \frac{3}{2} \end{cases}$$

BONUS: (5 Extra Points)

Plot $y(t)$ as a function of t with appropriate graph labelling. Use the graph below.

Solution:

The plot of $y(t)$ as a function of t is shown in the figure below.



Problem 3: Fourier Series with application for Parseval's Theorem (25 Points)

Consider the periodic continuous-time real signal $x(t)$ whose bandwidth does not exceed 50 Hz and whose single-sided magnitude and phase spectra are illustrated in Figure 1 below.

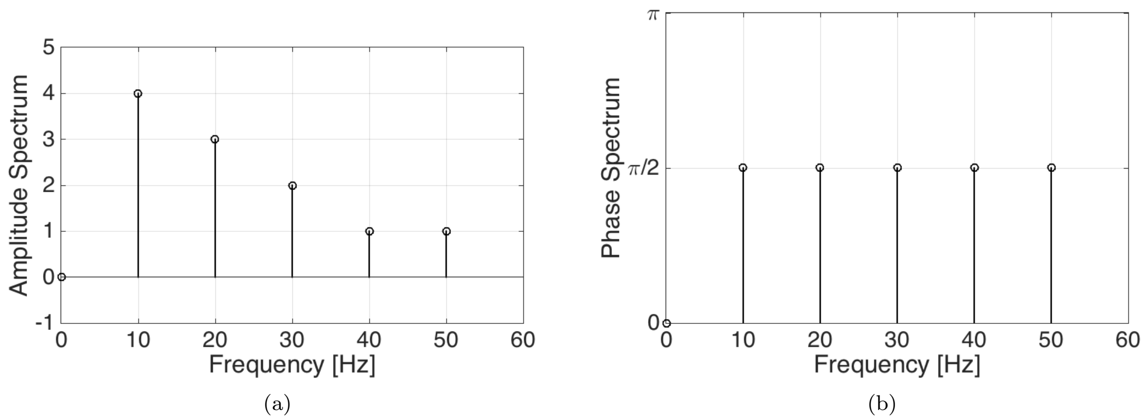


Figure 1: Amplitude and phase spectra of a signal $x(t)$.

Answer the following questions:

1. Find the fundamental frequency and the fundamental period of $x(t)$. **(2 Points)**
2. Determine the exponential Fourier Series coefficients of $x(t)$. **(5 Points)**
3. Determine whether $x(t)$ is even, odd or neither. Justify. **(2 Points)**
4. Determine the trigonometric Fourier Series coefficients of $x(t)$. **(5 Points)**
5. Compute the average value of $x(t)$. **(2 Points)**
6. Compute the total amount of power carried by $x(t)$. **(3 Points)**
7. Determine which of $x(t)$'s harmonics carry most of that power. **(2 Points)**
8. Compute the DC power and the power carried by $x(t)$'s third harmonic. **(4 Points)**

Solution:

1. Observe from the spectrum of the signal that two consecutive harmonics are separated by 10 Hz. Consequently:

$$f_0 = 10 \text{ (Hz)} \ \Rightarrow T_0 = \frac{1}{f_0} = 0.1 \text{ (s)}$$

2. From the single-sided spectrum of the signal, the following values of the exponential Fourier Series coefficients can be obtained:

$$\begin{aligned}
X[0] &= |X[0]|e^{j\arg(X[0])} = 0e^0 = 0 \\
X[1] &= |X[1]|e^{j\arg(X[1])} = 4e^{j\frac{\pi}{2}} = 4j \\
X[2] &= |X[2]|e^{j\arg(X[2])} = 3e^{j\frac{\pi}{2}} = 3j \\
X[3] &= |X[3]|e^{j\arg(X[3])} = 2e^{j\frac{\pi}{2}} = 2j \\
X[4] &= |X[4]|e^{j\arg(X[4])} = 1e^{j\frac{\pi}{2}} = j \\
X[5] &= |X[5]|e^{j\arg(X[5])} = 1e^{j\frac{\pi}{2}} = j \\
X[m] &= 0 \text{ , for } \forall m > 5
\end{aligned}$$

From the property $X[-m] = X^*[m]$, it can be concluded that:

$$X[-1] = -4j \quad ; \quad X[-2] = -3j \quad ; \quad X[-3] = -2j$$
$$X[-4] = -j \quad ; \quad X[-5] = -j \quad ; \quad X[-m] = 0, \text{ for } \forall m > 5$$

3. Recall that:

$$X[m] = \frac{X_c[m] - jX_s[m]}{2}$$

Having evaluated all the exponential Fourier Series coefficients above, observe that all of these coefficients are purely imaginary. As such, for all of these coefficients $X_c[m] = 0$, for $\forall m$. Hence, the signal is odd.

4. Since it was shown in question (3.) that $X_c[m] = 0$, $\forall m$ and having evaluated the exponential Fourier Series coefficients in question (1.), then the $X_s[m]$ can be evaluated using the relationship:

$$X[m] = \frac{X_c[m] - jX_s[m]}{2} = -j\frac{X_s[m]}{2} \Rightarrow X_s[m] = 2jX[m]$$

It follows that:

$$X_s[1] = -8 \quad ; \quad X_s[2] = -6 \quad ; \quad X_s[3] = -4$$
$$X_s[4] = -2 \quad ; \quad X_s[5] = -2 \quad ; \quad X_s[m] = 0, \text{ for } \forall m > 5$$

5. The average value of $x(t)$ is given by:

$$x_{\text{avg}} = \frac{1}{T_0}\int_{T_0} x(t)\mathrm{d}t = X[0] = 0$$

6. From Parseval's theorem, the total power carried by $x(t)$ is given by:

$$\begin{aligned} P_T &= |X[0]|^2 + 2\sum_{m\in\mathbb{Z}^+} |X[k]|^2 \\ &= |X[0]|^2 + 2[|X[1]|^2 + |X[2]|^2 + |X[3]|^2|X[4]|^2 + |X[5]|^2] \\ &= 0 + 2(16 + 9 + 4 + 1 + 1) = 62 \text{ (W)} \end{aligned}$$

7. From the magnitude spectrum, it is clear that the first harmonic (i.e. $m = 1$) carries the largest proportion of the power.

8. The DC power of $x(t)$ is given by:

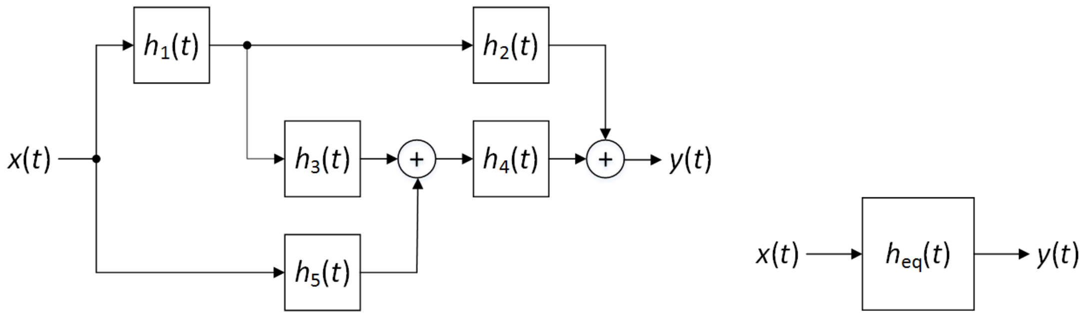
$$P_{DC} = |X[0]|^2 = 0 \text{ (W)}$$

The power carried by the third harmonic is given by:

$$P_3 = 2|X[3]|^2 = 2 \cdot 4 = 8 \text{ (W)}$$

Problem 4: Composite Impulse Response (25 Points)

Consider the Linear and Time-Invariant (LTI) system illustrated in figure 2(a).



(a) Original system. (b) Equivalent system.

Figure 2: LTI System.

This entire system is composed of several LTI subsystems represented by blocks that are annotated with these subsystems' individual impulse responses $h_1(t)$ through $h_5(t)$. These subsystems are to be combined into one single block whose equivalent impulse response is $h_{\text{eq}}(t)$.

1. Express $h_{\text{eq}}(t)$ as a function of $h_1(t)$, $h_2(t)$, $h_3(t)$, $h_4(t)$ and $h_5(t)$. **(10 Points)**
2. Assume that: **(15 Points)**

$$h_1(t) = h_4(t) = u(t)$$
$$h_2(t) = h_3(t) = 5\delta(t)$$
$$h_5(t) = e^{-2t}u(t)$$

   Derive an expression for $h_{\text{eq}}(t)$.

Solution:

1. Denote by $y_i(t)$ to be the output of subsystem i where $(i = 1, 2, 3, 4, 5)$. Therefore:

$$\begin{aligned}
y_1(t) &= x(t) * h_1(t) \\
y_2(t) &= y_1(t) * h_2(t) = [x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)] \\
y_3(t) &= y_1(t) * h_3(t) = [x(t) * h_1(t)] * h_3(t) = x(t) * [h_1(t) * h_3(t)] \\
y_5(t) &= x(t) * h_5(t) \\
y_4(t) &= [y_3(t) + y_5(t)] * h_4(t) \\
&= [x(t) * [h_1(t) * h_3(t)] + x(t) * h_5(t)] * h_4(t) \\
&= x(t) * [h_1(t) * h_3(t) * h_4(t) + h_5(t) * h_4(t)]
\end{aligned}$$

   It follows that, the overall output $y(t)$ is given by:

$$\begin{aligned}
y(t) &= y_4(t) + y_2(t) \\
&= x(t) * [h_1(t) * h_3(t) * h_4(t) + h_5(t) * h_4(t)] + x(t) * [h_1(t) * h_2(t)] \\
&= x(t) * [h_1(t) * h_3(t) * h_4(t) + h_4(t) * h_5(t) + h_1(t) * h_2(t)]
\end{aligned}$$

   Finally, the equivalent impulse response of the system is given by:

$$h_{\text{eq}}(t) = h_1(t) * h_3(t) * h_4(t) + h_4(t) * h_5(t) + h_1(t) * h_2(t)$$

2. Given the above assumed expressions for $h_1(t)$ through $h_5(t)$, then:

$$h_{\text{eq}}(t) = u(t) * 5\delta(t) * u(t) + u(t) * e^{-2t}u(t) + u(t) * 5\delta(t) = \left[5t + \frac{1}{2}(1 - e^{-2t}) + 5\right] u(t)$$